Determinants of Matrices

Determinants are a property of square matrices. The determinant of **A** is written $|A|$.

The determinant of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $|A| = a \cdot d - b \cdot c$

For higher rank matrices, we can use **cofactors** to calculate their determinants:

The cofactor of an element in row **I** and column **J** is the determinant of the matrix that remains after row **I** and column **J** are removed:

column 3

row 4 $\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}$

Here the cofactor for A_{43} is $\begin{vmatrix} A_{11} & A_{12} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{34} & A_{35} \\ A_{51} & A_{52} & A_{54} & A_{55} \end{vmatrix}$

To calculate the determinant of a 3×3 matrix:

$$\begin{matrix} + & - & + \end{matrix}$$
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Write alternating +'s and –'s above the top row

Multiply each element in the top row by its cofactor.

If a + is above the column, then add it.

If a –is above the column, then subtract it.

So $|A| = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$

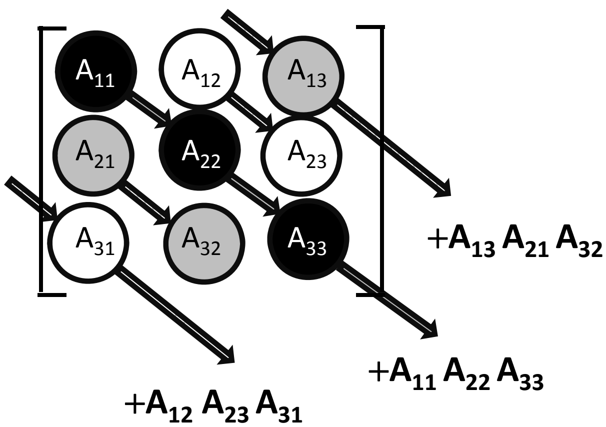
Or $|A| = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$

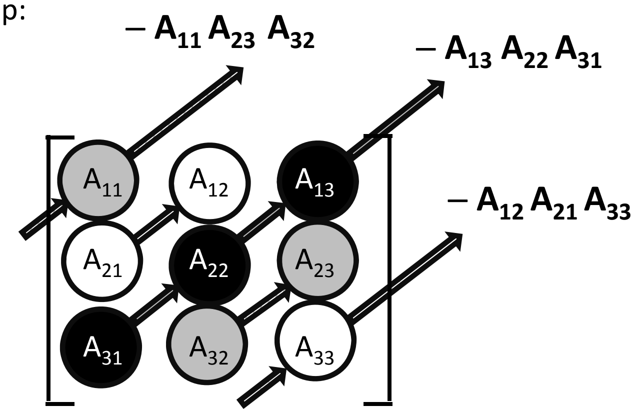
We can also write

$$|A| = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33} - A_{11}A_{23}A_{32}$$

Which we can also get by multiplying and adding all terms on lines starting at the upper left and going to the lower right, and then multiplying and subtracting all terms on lines starting at the lower left and going to the upper right.

A diagram may help:





Note: We can use the elements in any row to compute the determinant using cofactors.

Use a grid of alternating +'s and –'s to determine whether to a term should be added or subtracted

For example, given the 3 × 3 matrix **A** above, the grid would be $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$\text{So } |A| = -A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{22}\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} - A_{23}\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

$$= +A_{31}\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} - A_{32}\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} + A_{33}\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$